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SOME GENERALIZATIONS OF THE THEORY OF OPTIMIZATION OF CASCADE THERMOELECTRIC COOLING UNITS

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The optimal conditions are generalized, taking account of the irreversible losses accompanying thermoelectric cooling.

Earlier [1], the problem of optimization of a thermoelectric cooling unit was solved for the case when there are no irreversible losses associated with nonideality of the electrical contacts and external heat inflows. The conditions for a minimum of the functional $\mu = Q_h/Q_c$ or the equivalent additive functional

$$J = \sum_{k=1}^{N} \ln q_{1m}^{k} - \sum_{k=1}^{N} \ln q_{0m}^{k} \quad (1)$$

were obtained, taking account of the temperature dependence of the parameters of the thermoelectric materials.

The mathematical model proposed in [1] is now corrected for the case when the thermoelement junctions are of finite electrical resistance. The expressions for the heat-flux densities at the boundaries of the k-th cascade take the form

$$q_{0m}^{k} = 0.5\,[q_n(x_k^-) + q_p(x_k^-) + (i_n^k + i_p^k)\,R_c], \quad (2)$$

$$q_{1m}^{k} = 0.5\,[q_n(x_{k-1}^+) + q_p(x_{k-1}^+) - (i_n^k + i_p^k)\,R_c]. \quad (3)$$

All the remaining relations in [1] are retained, but the derivatives $\partial J/\partial i_n^{k,p}$ in the equations for the optimal current densities are no longer equal to zero. Taking account of Eqs. (1)-(3), it is found that in the general case

$$\frac{\partial J}{\partial i_{n,p}^{k}} = -\beta R_c \left(\frac{1}{q_{0m}^{k}} + \frac{1}{q_{1m}^{k}}\right), \quad (4)$$

where $\beta = 1$ when $s_n^k = s_p^k (i_n^k = i_p^k)$ and $\beta = 1/2$ when the parameters s_n and s_p vary independently.

The appropriate analysis shows that when $R_c \neq 0$ and there are no constraints on the longitudinal dimension of the apparatus, the coordinates x_k, k = 1, ..., N, must be indeterminably large, i.e., there is no optimal sequence l_k. In practice, this means that the lengths of all the thermoelements must be equal to some limitingly large dimensions compatible with the specified size of the device, and hence the condition in Eq. (14) of [1] may be eliminated, as before.

Now consider the case when there are heat inflows from the surrounding medium. Heat transfer occurs on the free part of the heat-transfer surface and on the side surfaces of the cascades. Below, no account is taken of the lateral heat inflows; instead, it is assumed

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that not part but all of the cold heat-transfer surface of each cascade participates in the heat exchange, as if the subsequent cascade were not present.* In each case, Eq. (2) may be brought to the form

$$q_{0m}^{k}=0.5\left[\,q_{n}(x_{\bar{k}})+q_{p}(x_{\bar{k}})+(i_{n}^{k}+i_{p}^{k})R_{c}+q_{T}^{k}\left(\frac{1}{i_{n}^{k}}+\frac{1}{i_{p}^{k}}\right)\right], \tag{5}$$

where

$$q_{T}^{k}=\alpha_{T}[T_{h}-T_{n}(x_{\bar{k}})],\ k=1,\ \ldots,\ N. \tag{5a}$$

Taking account of Eq. (5), the quantity $\mu=\prod_{k=1}^{N}(q_{1m}^{k}/q_{0m}^{k})$ retains its previous meaning of the ratio Q_h/Q_c, but the condition of a minimum of μ no longer corresponds to the minimum of the power at fixed Q_c, since the first law of thermodynamics requires that†

$$W=Q_{h}-Q_{c}-\sum_{k=1}^{N}Q_{T}^{k}=Q_{c}(\mu-1)-\sum_{k=1}^{N}Q_{T}^{k} \tag{6}$$

be written in place of the equality $W = Q_h - Q_c = Q_c(\mu - 1)$. It may be shown, however, that even in the given general case, μ may serve as the basic criterion for evaluating the efficiency of the cascade system. The condition of a minimum of this quantity corresponds to a minimum of Q_h at fixed Q_c. Thus, achieving a minimum of μ is an independent, practically important problem, corresponding to the widely encountered case where the possibility of heat output is limited. In addition, the conditions of a minimum of W and $\sum_{k=1}^{N}Q_{T}^{k}$, of which Q_h is composed, are noncontradictory, since all the measures leading to a reduction in the heat inflow also lead simultaneously to a decrease in the power for their compensation. Thus, it may be expected that the condition of a minimum is similar for all the terms in Eq. (6), so that, if a device with minimum Q_h is designed, it will be sufficiently close in its characteristics to a device with the minimum W (obviously the degree of deviation depends on the heat-transfer intensity; in the absence of heat inflows, the two variants coincide).

Mathematically, the specific features of the given problem are determined solely by the change in q_{0m}^{k} in accordance with Eq. (5). All the relations obtained in [1] are retained. Variations arise only in the expressions for the derivatives $\partial J/\partial i_{n,p}^{k}$ and $\partial J/\partial T_n(x_{\bar{k}})$ appearing in the optimal conditions [1]. Taking account of Eqs. (1), (3), and (5), it is found that

$$\frac{\partial J}{\partial i_{n,p}^{k}}=-\beta\left[R_{c}\left(\frac{1}{q_{0m}^{k}}+\frac{1}{q_{1m}^{k}}\right)-\frac{q_{T}^{k}}{q_{0m}^{k}(i_{n,p}^{k})^{2}}\right], \tag{7}$$

and the factor β has the same meaning as in Eq. (4). Further, since the boundary temperature $T_n(x_{\bar{k}})$ appears in Eq. (5), the derivative $\partial J/\partial T_n(x_{\bar{k}})$ is also nonzero. Taking account of Eqs. (1) and (5), it is found that

$$\frac{\partial J}{\partial T_{n}(x_{\bar{k}})}=\frac{\alpha_{T}}{2q_{0m}^{k}}\left(\frac{1}{i_{n}^{k}}+\frac{1}{i_{p}^{k}}\right),\ k=1,\ \ldots,\ N. \tag{8}$$

In combination with the conditions in Eqs. (11)-(15) of [1], Eqs. (7) and (8) obtained here give the solution of the optimal problem in the most general formulation. The variants considered earlier follow from these equations as particular cases for the assumptions $R_c = 0$ and (or) $\alpha_T = 0$.

The appropriate analysis shows that when $\alpha_T \neq 0$ and $R_c = 0$, the optimal variant corresponds to the conditions $x_k \to 0$, $k = 1, \ldots, N$. In practice, this means that all the thermo-

*This assumption is not obligatory. Other versions of the calculational model taking account of the lateral heat transfer are also possible. However, there is no need for this complication, since the estimates of the coefficients characterizing the heat transfer to individual sections of the surface are of low real accuracy.

†The terms of Q_T^k in Eq. (6) complicate the problem to an extreme degree. It is clear that the heat inflows depend on the size and number of junctions, which must now be included among the controlling parameters. In connection with this, the conditions of heat balance of the cascade junctions [1] may no longer be discarded in the course of optimization. In addition, the functional in Eq. (6) is not reduced to additive form. As a result, the condition of transversality and the condition of an optimum are complicated so much that the problem becomes practically unsolvable.

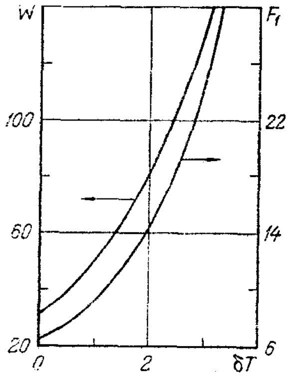

Fig. 1

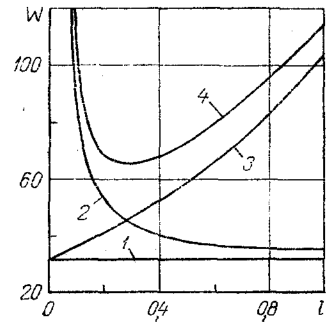

Fig. 2

Fig. 1. Dependence of the power and base area of the eight-cascade refrigeration unit on the cascade-junction temperature difference.

Fig. 2. Dependence of the power on the thermoelement length with various R_c ($\Omega\cdot cm^2$) and α_T ($W/m^2\cdot K$): 1) $R_c = 0$, $\alpha_T = 0$; 2) $R_c = 5\cdot10^{-6}$, $\alpha_T = 0$; 3) $R_c = 0$, $\alpha_T = 2$; 4) $R_c = 5\cdot10^{-6}$, $\alpha_T = 2$.

element lengths must be taken to be equal to some limitingly small dimension, which is specified on the basis of technological considerations or from constraints imposed on the parameters of the supply. Thus, in this particular case, as well, l_k may be regarded as specified and hence Eq. (14) in [1] may be eliminated. In the general case ($\alpha_T \neq 0$, $R_c \neq 0$), there are contradictory requirements on the ratio l_k; this leads to the existance of an optimal sequence of thermoelement lengths. To determine this sequence, the optimal conditions must include Eq. (14) of [1] if, ultimately, the dimensions l_k are not fixed by the conditions of the problem. Note that the method of successive approximation described in [1] may be used to solve the generalized optimal problem without any modifications.

The development of a numerical-optimization program allows a large number of variants of cascade devices for the most diverse conditions to be accurately calculated. This offers the possibility of establishing the basic trends in the behavior of the optimal parameters and estimating the reserves for increase in efficiency of low-temperature cascade systems. Data for an eight-cascade device obtained in [1] without taking account of the contact resistances and external heat transfer are taken as the control variant for estimating the influence of various factors. In accordance with [2], the following input parameters are fixed here: $T_h = 325°K$, $T_c = 145°K$, $Q_c = 10$ mW, $l_k = l = 0.25$ cm, $k = 1, \ldots, 8$. In all the calculations, the same data on the temperture dependence of the properties of the semiconductor materials as for the control variant are used [1].

The calculation results given below illustrate the high sensitivity of the cascade system to various perturbations, which confirms theneed to use rigorous optimization methods.

Influence of Contact Electric Resistances and Intercascade Temperature Differences

In single-cascade thermobatteries, contact resistance at the hot junctions leads to increase in power but does not reduce the refrigeration capacity. In a multicascade system, contact resistances on both sides of the cascade boundary lead simultaneously to increase in power and to reduction in cooling effect. In connection with this, a more considerable influence of the contact resistances on the indices of the cascade systems is expected.

Below, values of W and F_1 are given for a series of values of R_c

$R_c = 0$,	$W = 31.7$,	$F_1 = 6.09$,
10^{-6},	34.2,	6.55,
$5\cdot10^{-6}$,	46.6,	8.76,
10^{-5},	69.2,	12.71,
$5\cdot10^{-5}$,	2538.1,	399.12.

With $R_c = 10^{-5}$ $\Omega\cdot cm^2$, which is regarded as sufficiently low, the power and dimensions of the refrigeration unit increase more than twofold in comparison with the control variant. This result indicates the need for careful commutation of the cascade thermobatteries.

TABLE 1. Influence of External Heat Inflows ($N = 8$, $T_h = 325°K$, $T_c = 145°K$, $Q_c = 10$ mW, $l = 0.25$ cm)

Variant No.	Heat transfer conditions	$F_0 = F_N$		$F_0 = 0.5$ cm^2	
		W, W	base area F_1, cm^2	W, W	base area F_1, cm^2
1	Adiabatic insulation ($\alpha_T = 0$)	31,70	6,09	31,70	6,09
2	Radiant heat transfer	43,46	8,28	173,54	33,11
3	Convection in vacuum at residual pressures:				
	10^{-6} torr ($\alpha_T = 0{,}07$ W/m^2·K)	32,06	6,16	35,97	6,91
	10^{-5} ($\alpha_T = 0{,}64$)	35,06	6,71	71,75	13,72
	10^{-4} ($\alpha_T = 1{,}67$)	41,08	7,85	144,39	27,61
	10^{-3} ($\alpha_T = 3{,}64$)	55,06	10,81	320,93	62,52
	10^{-2} ($\alpha_T = 6{,}75$)	85,68	18,08	729,10	154,05

Temperature differences at the cascade junctions also exert a significant influence (Fig. 1). With δT equal to no more than 1°K, W increases by 58% in comparison with the control variant. Further increase in δT leads to still sharper increase in the power and dimensions. Note that in Fig. 1 (and the other figures), each point on the abscissa corresponds to a particular constructional variant of the cascade unit which is optimal for the given value of the independent variable.

Influence of External Heat Inflows

Table 1 gives the values of W and F_1 for various heat-transfer conditions of a refrigeration unit with the external surroundings (the values of α_T are obtained by analysis of the data in [3]). As is evident from Table 1, the cascade system exhibits considerable sensitivity to external heat inflows. Radiant heat transfer exerts a significant influence: even when the area of the cooled substrate F_0 is no more than the area of the low-temperature cascade, the power must be increased by 37% in comparison with the variant with adiabatic insulation. Thus, the need for screening and the maintenance of high vacuum is obvious. It also follows from Table 1 that adding a substrate of greater size to the low-temperature cascade may result in multiple increase of power and size of the device. This must be taken into account in the construction of low-temperature cascade systems.

Influence of Thermoelement Height

The dependence of W on the thermoelement height with various combinations of the parameters R_c and α_T is shown in Fig. 2. The horizontal straight line 1 corresponds to the control variant ($R_c = 0$, $\alpha_T = 0$). As is evident from Fig. 2, when $\alpha_T = 0$ and $R_c \neq 0$ (curve 2), there is no optimal height: monotonic reduction in W with increase in l is observed. When $R_c = 0$ and $\alpha_T \neq 0$ (curve 3), the opposite trend is observed: W increases monotonically with increase in l. The combined influence of the contact resistance and heat transfer leads to the existence of an optimal thermoelement height (curve 4); the optimum is shifted toward larger l with increase in R_c (Fig. 3). Note also the considerable sensitivity of the system to leftward deviation from the extremal point.

Influence of the Quality of the Initial Material

It is of considerable interest to establish how sensitive the cascade system is to the quality of the initial semiconducting material. To this end, calculations are performed for a series of variants, when $z(T)$ for thermoelements of both types is reduced with respect to the control material by 5, 10, 15, and 20%, over the whole temperature range. The results of the calculation are shown in Fig. 4 in the form of a dependence of W on γ, where γ is the coefficient of Q-factor reduction. As is evident from Fig. 4, reduction in z by no more than 15% results in a fourfold deterioration in the energy indices in comparison with the control variant. Note that the parameters of the semiconductors are often measured with an error of the order of 5-10%, which is regarded as satisfactory for a single-cascade unit. The situation is completely different when the optimization of multicascade systems is considered. Rigorous theory directly rules out the possibility, in principle, of obtaining reliable results if the input data are measured with the given error. Thus, the use of an accurate theoretical method must be combined with precision measurements of the thermoelectric parameters.

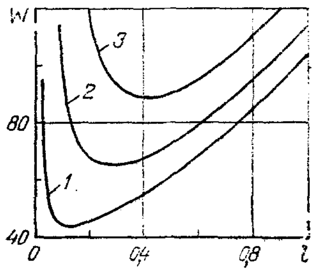

Fig. 3

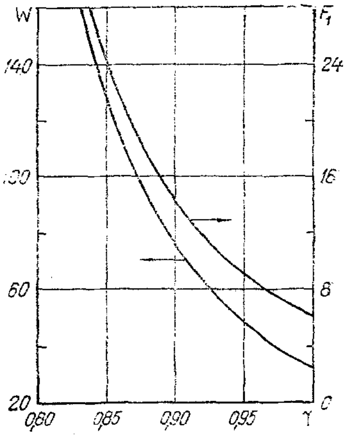

Fig. 4

Fig. 3. Dependence of the power on the thermoelement length with $\alpha_T = 2\ W/m^2 \cdot K$ which corresponds to the conditions of radiant heat transfer for R_c ($\Omega \cdot cm^2$): 1) 10^{-6}; 2) $5 \cdot 10^{-6}$; 3) 10^{-5}.

Fig. 4. Dependence of the power and base area of an eight-cascade refrigeration unit on the coefficient of thermoelectric Q-factor reduction.

TABLE 2. Prospects for Reducing the Power W (W) by Using Bi—Sb Alloy ($N = 8$, $T_h = 325°K$, $T_c = 145°K$, $Q_c = 10$ mW)

Number of low temperature cascades in which Bi—Sb alloy is used in the n branch	0	1	2	3	4	5	6	7	8
Cross-section ratio is not optimized ($S_n = S_p$)	31,7	28,4	28,2	30,1	34,6				
Cross-section ratio is optimized	29,2	20,4	16,6	14,7	13,7	13,5	13,7	14,6	16,2

Prospects for Using the Bi—Sb System in Low-Temperature Cascades

It is known that at relatively low temperatures, Bi—Sb alloys are more effective than thermoelectric materials based on Bi_2Te_3. In connection with this, calculations have been performed to evaluate the possibility of reducing the cascade-system power when the usual materials are replaced by an n-type material based on Bi—Sb in low-temperature cascades. In the calculations, data on the properties of the semiconducting Bi—Sb alloy with 8% bismuth are used [4].

The first series of calculations is for the case when the cross-sectional area of the p-type and n-type thermoelements is the same. The data are shown in the first row of Table 2, from which it is evident that the best result is obtained using the Bi—Sb alloy in two low-temperature cascades. The reduction in power is small — of the order of 11%. Using the Bi—Sb alloy in more than two low-temperature cascades is inexpedient, since W increases here.

An incomparably large effect is obtained when the constraint $s_n = s_p$ is eliminated, in which case, together with the optimization with respect to the current, optimization of the ratio of junction areas for the n-type and p-type branches is necessary. The results are shown in the second row of Table 2, from which it is evident that the use of Bi—Sb alloy only in the first low-temperature cascade reduces the specific power to 20.4 W, i.e., by 35%. Further substitution of material in the n branch amplifies the effect. When Bi—Sb alloy is used in five low-temperature cascades, the specific power is reduced to 13.5 W, i.e., by more than half. Thus, there is a considerable reserve for the increase in efficiency of cascade refrigeration units, which may be realized by using Bi—Sb alloys under the condition of correct matching of the branch cross sections.

NOTATION

N, number of cascades; ρ, electrical resistivity, $\Omega\cdot$cm; z, thermoelectric Q-factor parameter, $°K^{-1}$; x, coordinate, cm; $y(x_k^-)$, $y(x_k^+)$, values of the function y(x) immediately to the left and the right of the point x_k, respectively; T, absolute temperature, °K; T_h, T_c, temperatures of the heat-scattering and heat-receptive faces of the cascade unit, °K; δT, temperature jump at the cascade junction, °K; i, electric current density, A/cm^2; q = Q/I, specific heat flux, V; Q, total heat flux in the thermoelement cross section, W; I, electric current, A; q_{om}^k, q_{im}^k, mean specific heat fluxes at the cold and hot faces of the k-th cascade, V; Q_c, heat liberated from the object being cooled, W; Q_h, heat transmitted to the surrounding medium, W; $\mu = Q_h/Q_c$; J = ln μ, functional to be minimized; W, power, W; R_c, electrical resistance of unit area of contact, $\Omega\cdot cm^2$; α_T, heat-transfer coefficient, $W/m^2\cdot K$; Q_T^k, heat inflow to cold face of k-th cascade from the surrounding medium, W; s, junction area, cm^2; l, thermoelement length, cm; F_0, area of external face of substrate being cooled, cm^2; γ, Q-factor reduction parameter. Indices: n, p, electron, hole thermoelement; k, number of cascade.

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NUMERICAL DETERMINATION OF TWO-DIMENSIONAL TEMPERATURE FIELDS IN TRANSPIRATION COOLING

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We present a numerical method for solving a wide range of transpiration cooling problems.

It is well known that an analytic calculation of one-dimensional temperature fields in transpiration cooling of a plate with any (linear) boundary conditions reduces to a sequence of standard mathematical operations, and presents no difficulties. However, even for a one-dimensional problem of the transpiration cooling of a cylinder, and especially for two-dimensional problems, analytic solutions exist only in rare special cases. Accordingly, it is clear that a more general study of transpiration cooling processes must be based on a numerical solution of the appropriate equations.

We know of only one paper devoted to a discussion of the numerical solution of a two-dimensional transpiration cooling problem [1]. In constructing an algorithm Koh and Colony [1] employed a discrete Fourier transform. In doing this they essentially used the simplicity of the eigenfunctions of a difference operator along one of the directions. A rather wide range of problems can be solved by using the algorithm described in [1]. The optimum use of the Fourier transform described, for example, in [2] enables one to obtain a rather economical machine realization of the algorithm. However, the algorithm proposed by Koh and Colony [1] has shortcomings. In particular, it cannot be used to solve problems: 1) of once-through cooling of a porous cylinder; 2) with boundary conditions of the third kind along im-

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